

ON HEREDITARY PROPERTIES OF QUANTUM GROUP AMENABILITY

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ABSTRACT. Given a locally compact quantum group \mathbb{G} and a closed quantum subgroup \mathbb{H} , we show that \mathbb{G} is amenable if and only if \mathbb{H} is amenable and \mathbb{G} acts amenably on the quantum homogenous space \mathbb{G}/\mathbb{H} . We also study the existence of $L^1(\widehat{\mathbb{G}})$ -module projections from $L^\infty(\widehat{\mathbb{G}})$ onto $L^\infty(\widehat{\mathbb{H}})$.

1. INTRODUCTION

Let G be a locally compact group. If G is amenable, it is well-known that any closed subgroup H of G is amenable and that G acts amenably on the homogenous space G/H in the sense that there is a G -invariant state $m \in L^\infty(G/H)^*$. Conversely, if H is an amenable closed subgroup of G for which G acts amenably on G/H , then G is necessarily amenable. We show that this fundamental hereditary property of amenability persists at the level of locally compact quantum groups using a recent homological characterization amenability [3, Theorem 5.2], which states that a locally compact quantum group \mathbb{G} is amenable if and only if its dual $L^\infty(\widehat{\mathbb{G}})$ is 1-injective as an operator module over $L^1(\widehat{\mathbb{G}})$.

The existence of (completely) bounded $A(G)$ -module projections $P : VN(G) \rightarrow VN(H)$ has been a recent topic of interest in harmonic analysis [8, 9, 12, 15]. In particular, it was shown in [9, Theorem 12] that if H is amenable, then a bounded $A(G)$ -module projection always exists. We show that this projection may be taken completely contractive, and, in addition, that such a projection exists for a large class of locally compact quantum groups.

2. PRELIMINARIES

Let \mathcal{A} be a complete contractive Banach algebra. We say that an operator space X is a right *operator \mathcal{A} -module* if it is a right Banach \mathcal{A} -module such that the module map $m_X : X \widehat{\otimes} \mathcal{A} \rightarrow X$ is completely contractive, where $\widehat{\otimes}$ denotes the operator space projective tensor product. We say that X is *faithful* if for every non-zero $x \in X$, there is $a \in \mathcal{A}$ such that $x \cdot a \neq 0$. We denote by $\mathbf{mod} - \mathcal{A}$ the category of right operator \mathcal{A} -modules with morphisms given by completely bounded module homomorphisms. Left operator \mathcal{A} -modules and operator \mathcal{A} -bimodules are defined similarly, and we denote the respective categories by $\mathcal{A} - \mathbf{mod}$ and $\mathcal{A} - \mathbf{mod} - \mathcal{A}$.

Let \mathcal{A} be a completely contractive Banach algebra and $X \in \mathbf{mod} - \mathcal{A}$. The identification $\mathcal{A}^+ = \mathcal{A} \oplus_1 \mathbb{C}$ turns the unitization of \mathcal{A} into a unital completely

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contractive Banach algebra, and it follows that X becomes a right operator \mathcal{A}^+ -module via the extended action

$$x \cdot (a + \lambda e) = x \cdot a + \lambda x, \quad a \in \mathcal{A}^+, \lambda \in \mathbb{C}, x \in X.$$

There is a canonical completely contractive morphism $\Delta^+ : X \rightarrow \mathcal{CB}(\mathcal{A}^+, X)$ given by

$$\Delta^+(x)(a) = x \cdot a, \quad x \in X, a \in \mathcal{A}^+,$$

where the right \mathcal{A} -module structure on $\mathcal{CB}(\mathcal{A}^+, X)$ is defined by

$$(\Psi \cdot a)(b) = \Psi(ab), \quad a \in \mathcal{A}, \Psi \in \mathcal{CB}(\mathcal{A}^+, X), b \in \mathcal{A}^+.$$

An analogous construction exists for objects in $\mathcal{A} - \mathbf{mod}$. Let $C \geq 1$. We say that X is *relatively C -injective* if there exists a morphism $\Phi^+ : \mathcal{CB}(\mathcal{A}^+, X) \rightarrow X$ such that $\Phi^+ \circ \Delta^+ = \text{id}_X$ and $\|\Phi^+\|_{cb} \leq C$. When X is faithful, this is equivalent to the existence a morphism $\Phi : \mathcal{CB}(\mathcal{A}, X) \rightarrow X$ such that $\Phi \circ \Delta = \text{id}_X$ and $\|\Phi\|_{cb} \leq C$ by the operator analogue of [5, Proposition 1.7], where $\Delta(x)(a) := \Delta^+(x)(a)$ for $x \in X$ and $a \in \mathcal{A}$. We say that X is *C -injective* if for every $Y, Z \in \mathbf{mod} - \mathcal{A}$, every completely isometric morphism $\Psi : Y \hookrightarrow Z$, and every morphism $\Phi : Y \rightarrow X$, there exists a morphism $\tilde{\Phi} : Z \rightarrow X$ such that $\|\tilde{\Phi}\|_{cb} \leq C\|\Phi\|_{cb}$ and $\tilde{\Phi} \circ \Psi = \Phi$.

A *locally compact quantum group* is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a Hopf-von Neumann algebra with co-multiplication $\Gamma : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$, and φ and ψ are fixed left and right Haar weights on $L^\infty(\mathbb{G})$, respectively [17, 19]. For every locally compact quantum group \mathbb{G} , there exists a *left fundamental unitary operator* W on $L_2(\mathbb{G}, \varphi) \otimes L_2(\mathbb{G}, \varphi)$ and a *right fundamental unitary operator* V on $L_2(\mathbb{G}, \psi) \otimes L_2(\mathbb{G}, \psi)$ implementing the co-multiplication Γ via

$$\Gamma(x) = W^*(1 \otimes x)W = V(x \otimes 1)V^*, \quad x \in L^\infty(\mathbb{G}).$$

Both unitaries satisfy the *pentagonal relation*; that is,

$$(2.1) \quad W_{12}W_{13}W_{23} = W_{23}W_{12} \quad \text{and} \quad V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

By [17, Proposition 2.11], we may identify $L_2(\mathbb{G}, \varphi)$ and $L_2(\mathbb{G}, \psi)$, so we will simply use $L^2(\mathbb{G})$ for this Hilbert space throughout the paper. We denote by R the unitary antipode of \mathbb{G} .

Let $L^1(\mathbb{G})$ denote the predual of $L^\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces an associative completely contractive multiplication on $L^1(\mathbb{G})$, defined by

$$\star : L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \ni f \otimes g \mapsto f \star g = \Gamma_*(f \otimes g) \in L^1(\mathbb{G}).$$

The canonical $L^1(\mathbb{G})$ -bimodule structure on $L^\infty(\mathbb{G})$ is given by

$$f \star x = (\text{id} \otimes f)\Gamma(x), \quad x \star f = (f \otimes \text{id})\Gamma(x), \quad x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G}).$$

A *left invariant mean* on $L^\infty(\mathbb{G})$ is a state $m \in L^\infty(\mathbb{G})^*$ satisfying

$$(2.2) \quad \langle m, x \star f \rangle = \langle f, 1 \rangle \langle m, x \rangle, \quad x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G}).$$

Right and two-sided invariant means are defined similarly. A locally compact quantum group \mathbb{G} is said to be *amenable* if there exists a left invariant mean on $L^\infty(\mathbb{G})$. It is known that \mathbb{G} is amenable if and only if there exists a right (equivalently, two-sided) invariant mean (cf. [10, Proposition 3]). We say that \mathbb{G} is *co-amenable* if $L^1(\mathbb{G})$ has a bounded left (equivalently, right or two-sided) approximate identity (cf. [1, Theorem 3.1]).

The *left regular representation* $\lambda : L^1(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ of \mathbb{G} is defined by

$$\lambda(f) = (f \otimes \text{id})(W), \quad f \in L^1(\mathbb{G}),$$

and is an injective, completely contractive homomorphism from $L^1(\mathbb{G})$ into $\mathcal{B}(L^2(\mathbb{G}))$. Then $L^\infty(\widehat{\mathbb{G}}) := \{\lambda(f) : f \in L^1(\mathbb{G})\}''$ is the von Neumann algebra associated with the dual quantum group $\widehat{\mathbb{G}}$. Analogously, we have the *right regular representation* $\rho : L^1(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ defined by

$$\rho(f) = (\text{id} \otimes f)(V), \quad f \in L^1(\mathbb{G}),$$

which is also an injective, completely contractive homomorphism from $L^1(\mathbb{G})$ into $\mathcal{B}(L^2(\mathbb{G}))$. Then $L^\infty(\widehat{\mathbb{G}}') := \{\rho(f) : f \in L^1(\mathbb{G})\}''$ is the von Neumann algebra associated to the quantum group $\widehat{\mathbb{G}}'$. It follows that $L^\infty(\widehat{\mathbb{G}}') = L^\infty(\widehat{\mathbb{G}})'$, and the left and right fundamental unitaries satisfy $W \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ and $V \in L^\infty(\widehat{\mathbb{G}}') \overline{\otimes} L^\infty(\mathbb{G})$ [17, Proposition 2.15]. Moreover, dual quantum groups always satisfy $L^\infty(\mathbb{G}) \cap L^\infty(\widehat{\mathbb{G}}) = L^\infty(\mathbb{G}) \cap L^\infty(\widehat{\mathbb{G}}') = \mathbb{C}1$ [21, Proposition 3.4].

If G is a locally compact group, we let $\mathbb{G}_a = (L^\infty(G), \Gamma_a, \varphi_a, \psi_a)$ denote the *co-commutative* quantum group associated with the commutative von Neumann algebra $L^\infty(G)$, where the co-multiplication is given by $\Gamma_a(f)(s, t) = f(st)$, and φ_a and ψ_a are integration with respect to a left and right Haar measure, respectively. The dual $\widehat{\mathbb{G}}_a$ of \mathbb{G}_a is the *co-commutative* quantum group $\mathbb{G}_s = (VN(G), \Gamma_s, \varphi_s, \psi_s)$, where $VN(G)$ is the left group von Neumann algebra with co-multiplication $\Gamma_s(\lambda(t)) = \lambda(t) \otimes \lambda(t)$, and $\varphi_s = \psi_s$ is Haagerup's Plancherel weight. Then $L^1(\mathbb{G}_a)$ is the usual group convolution algebra $L^1(G)$, and $L^1(\mathbb{G}_s)$ is the Fourier algebra $A(G)$. A commutative quantum group is amenable if and only if its underlying locally compact group is amenable, while co-commutative quantum groups are always amenable.

For general locally compact quantum groups \mathbb{G} , the left and right fundamental unitaries yield canonical extensions of the co-multiplication given by

$$\Gamma^l : \mathcal{B}(L^2(\mathbb{G})) \ni T \mapsto W^*(1 \otimes T)W \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}));$$

$$\Gamma^r : \mathcal{B}(L^2(\mathbb{G})) \ni T \mapsto V(T \otimes 1)V^* \in \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} L^\infty(\mathbb{G}).$$

In turn, we obtain a canonical $L^1(\mathbb{G})$ -bimodule structure on $\mathcal{B}(L^2(\mathbb{G}))$ with respect to the actions

$$f \triangleright T = (\text{id} \otimes f)\Gamma^r(T), \quad T \triangleleft f = (f \otimes \text{id})\Gamma^l(T) \quad f \in L^1(\mathbb{G}), \quad T \in \mathcal{B}(L^2(\mathbb{G})).$$

3. CLOSED QUANTUM SUBGROUPS AND AMENABILITY

Let \mathbb{G} and \mathbb{H} be two locally compact quantum groups. Then \mathbb{H} is said to be a *closed quantum subgroup of \mathbb{G} in the sense of Vaes* if there exists a normal, unital, injective *-homomorphism $\gamma : L^\infty(\widehat{\mathbb{H}}) \rightarrow L^\infty(\widehat{\mathbb{G}})$ satisfying

$$(3.1) \quad (\gamma \otimes \gamma) \circ \Gamma_{\widehat{\mathbb{H}}} = \Gamma_{\widehat{\mathbb{G}}} \circ \gamma.$$

This is not the original definition of Vaes [20, Definition 2.5], but was shown to be equivalent in [7, Theorem 3.3]. With this definition, we have an analog of the Herz restriction theorem [13] for quantum groups, that is, $\gamma_* : L^1(\widehat{\mathbb{G}}) \rightarrow L^1(\widehat{\mathbb{H}})$ is a complete quotient map [7, Theorem 3.7]. Indeed, if \mathbb{G} and \mathbb{H} are commutative, with underlying locally compact groups G and H , the map γ is nothing but the canonical inclusion

$$VN(H) \ni \lambda_H(s) \mapsto \lambda_G(s) \in VN(G),$$

where λ_H and λ_G are the left regular representations of H and G , respectively. Its pre-adjoint γ_* is then the canonical quotient map $A(G) \rightarrow A(H)$ arising from the classical Herz restriction theorem.

Remark 3.1. There is an a priori weaker notion of closed quantum subgroup of a locally compact quantum group due to Woronowicz [7, Definition 3.2]. In what follows, we restrict ourselves to Vaes' definition, so that a *closed quantum subgroup* of a locally compact quantum group will always refer to the definition (3.1) given above.

Given a closed quantum subgroup \mathbb{H} of a locally compact quantum group \mathbb{G} , we let $L^\infty(\mathbb{G}/\mathbb{H}) := \gamma(L^\infty(\widehat{\mathbb{H}}))' \cap L^\infty(\mathbb{G})$ represent the quantum homogenous space \mathbb{G}/\mathbb{H} . By the left version of [16, Proposition 3.5] it follows that $\Gamma(L^\infty(\mathbb{G}/\mathbb{H})) \subseteq L^\infty(\mathbb{G}/\mathbb{H}) \overline{\otimes} L^\infty(\mathbb{G})$. Hence, $L^\infty(\mathbb{G}/\mathbb{H})$ is a left operator $L^1(\mathbb{G})$ -submodule of $L^\infty(\mathbb{G})$ under the action

$$f \star x = (\text{id} \otimes f)\Gamma(x), \quad x \in L^\infty(\mathbb{G}/\mathbb{H}), \quad f \in L^1(\mathbb{G}).$$

We say that \mathbb{G} *acts amenably on* \mathbb{G}/\mathbb{H} if there exists a state $m \in L^\infty(\mathbb{G}/\mathbb{H})^*$ satisfying

$$\langle m, f \star x \rangle = \langle f, 1 \rangle \langle m, x \rangle, \quad x \in L^\infty(\mathbb{G}/\mathbb{H}), \quad f \in L^1(\mathbb{G}).$$

Let $W^\mathbb{H} \in L^\infty(\mathbb{H}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}})$ be the left fundamental unitary of \mathbb{H} . By the left version of [7, Theorem 3.3] (see also [16, Lemma 3.11]) it follows that the unitary $(\text{id} \otimes \gamma)(W^\mathbb{H}) \in L^\infty(\mathbb{H}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ defines a co-action of \mathbb{H} on \mathbb{G} by

$$\alpha : L^\infty(\mathbb{G}) \ni x \mapsto (\text{id} \otimes \gamma)(W^\mathbb{H})^* (1_{L^\infty(\mathbb{H})} \otimes x) (\text{id} \otimes \gamma)(W^\mathbb{H}) \in L^\infty(\mathbb{H}) \overline{\otimes} L^\infty(\mathbb{G}),$$

where α satisfies $(\text{id} \otimes \alpha) \circ \alpha = (\Gamma_\mathbb{H} \otimes \text{id}) \circ \alpha$. This co-action defines a right operator $L^1(\mathbb{H})$ -module structure on $L^\infty(\mathbb{G})$ by

$$x \star_\mathbb{H} g = (g \otimes \text{id})\alpha(x), \quad x \in L^\infty(\mathbb{G}), \quad g \in L^1(\mathbb{H}).$$

Moreover, the quantum homogenous space $L^\infty(\mathbb{G}/\mathbb{H})$ is precisely the fixed point algebra $L^\infty(\mathbb{G})^\alpha = \{x \in L^\infty(\mathbb{G}) \mid \alpha(x) = 1 \otimes x\}$ [16, Lemma 3.11]. Let $\alpha^l : \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{H}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))$ denote the canonical extension of α , given by

$$\alpha^l(T) = (\text{id} \otimes \gamma)(W^\mathbb{H})^* (1_{L^\infty(\mathbb{H})} \otimes T) (\text{id} \otimes \gamma)(W^\mathbb{H}), \quad T \in \mathcal{B}(L^2(\mathbb{G})).$$

We then obtain an extended right $L^1(\mathbb{H})$ -module action on $\mathcal{B}(L^2(\mathbb{G}))$.

Theorem 3.2. *Let \mathbb{G} and \mathbb{H} be two locally compact quantum groups such that \mathbb{H} is a closed quantum subgroup of \mathbb{G} . Then \mathbb{G} is amenable if and only if \mathbb{H} is amenable and \mathbb{G} acts amenably on \mathbb{G}/\mathbb{H} .*

Proof. If \mathbb{H} is amenable and \mathbb{G} acts amenably on \mathbb{G}/\mathbb{H} , there are states $m_\mathbb{H} \in L^\infty(\mathbb{H})^*$ and $m_\mathbb{G} \in L^\infty(\mathbb{G}/\mathbb{H})^*$ which are invariant under the canonical $L^1(\mathbb{H})$ and $L^1(\mathbb{G})$ actions on $L^\infty(\mathbb{H})$ and $L^\infty(\mathbb{G}/\mathbb{H})$, respectively. In particular, $m_\mathbb{H}$ satisfies

$$(m_\mathbb{H} \otimes \text{id})\Gamma_\mathbb{H}(y) = \langle m_\mathbb{H}, y \rangle 1, \quad y \in L^\infty(\mathbb{H}).$$

Define the completely positive map $P : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ by

$$P(x) = (m_\mathbb{H} \otimes \text{id})\alpha(x), \quad x \in L^\infty(\mathbb{G}).$$

Then, on the one hand,

$$\begin{aligned}\alpha(P(x)) &= \alpha((m_{\mathbb{H}} \otimes \text{id})\alpha(x)) = (m_{\mathbb{H}} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \alpha)\alpha(x) \\ &= (m_{\mathbb{H}} \otimes \text{id} \otimes \text{id})(\Gamma_{\mathbb{H}} \otimes \text{id})\alpha(x) = 1 \otimes ((m_{\mathbb{H}} \otimes \text{id})\alpha(x)) \\ &= 1 \otimes P(x),\end{aligned}$$

so $P(x) \in L^\infty(\mathbb{G}/\mathbb{H})$. On the other hand, since $(\text{id} \otimes \gamma)(W^{\mathbb{H}}) \in L^\infty(\mathbb{H}) \overline{\otimes} \gamma(L^\infty(\widehat{\mathbb{G}}))$, it follows that $P(axb) = aP(x)b$ for all $a, b \in L^\infty(\mathbb{G}/\mathbb{H}) = \gamma(L^\infty(\widehat{\mathbb{H}}))' \cap L^\infty(\mathbb{G})$. Thus, $P : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ is a completely positive projection onto $L^\infty(\mathbb{G}/\mathbb{H})$. Moreover, as $V \in L^\infty(\widehat{\mathbb{G}}') \overline{\otimes} L^\infty(\mathbb{G})$, we have $(\text{id} \otimes \gamma)(W^{\mathbb{H}})_{12} V_{23} = V_{23}(\text{id} \otimes \gamma)(W^{\mathbb{H}})_{12}$, so that

$$\begin{aligned}P(f \star_{\mathbb{G}} x) &= (m_{\mathbb{H}} \otimes \text{id})\alpha((\text{id} \otimes f)V(x \otimes 1)V^*) \\ &= (m_{\mathbb{H}} \otimes \text{id})(\text{id} \otimes \text{id} \otimes f)(\alpha \otimes \text{id})(V(x \otimes 1)V^*) \\ &= (m_{\mathbb{H}} \otimes \text{id})(\text{id} \otimes \text{id} \otimes f)((\text{id} \otimes \gamma)(W^{\mathbb{H}})_{12}^* V_{23}(x \otimes 1)_{23} V_{23}^*(\text{id} \otimes \gamma)(W^{\mathbb{H}})_{12}) \\ &= (m_{\mathbb{H}} \otimes \text{id})(\text{id} \otimes \text{id} \otimes f)(V_{23}(\text{id} \otimes \gamma)(W^{\mathbb{H}})_{12}^*(x \otimes 1)_{23}(\text{id} \otimes \gamma)(W^{\mathbb{H}})_{12} V_{23}^*) \\ &= (m_{\mathbb{H}} \otimes \text{id})(\text{id} \otimes \text{id} \otimes f)(V_{23}(\alpha(x) \otimes 1)V_{23}^*) \\ &= (\text{id} \otimes f)(V(P(x) \otimes 1)V^*) \\ &= f \star_{\mathbb{G}} P(x),\end{aligned}$$

for all $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$. Thus, P is a left $L^1(\mathbb{G})$ -module map. Defining $m := m_{\mathbb{G}} \circ P \in L^\infty(\mathbb{G})^*$, we obtain a right invariant mean on $L^\infty(\mathbb{G})$, whence \mathbb{G} is amenable.

Conversely, suppose that \mathbb{G} is amenable. Clearly the restriction of a right invariant mean to $L^\infty(\mathbb{G}/\mathbb{H})$ will be $L^1(\mathbb{G})$ -invariant, so \mathbb{G} acts amenably on \mathbb{G}/\mathbb{H} . We show that \mathbb{H} is amenable.

By [3, Theorem 5.2] $L^\infty(\widehat{\mathbb{G}})$ is 1-injective in $L^1(\widehat{\mathbb{G}})$ -**mod**. The space $\mathcal{B}(L^2(\mathbb{H})) = \mathcal{B}(L^2(\widehat{\mathbb{H}}))$ becomes a left operator $L^1(\widehat{\mathbb{G}})$ -module via:

$$(3.2) \quad \hat{f} \triangleright_{\widehat{\mathbb{G}}} T = (\text{id} \otimes \gamma_*(\hat{f}))V^{\widehat{\mathbb{H}}}(T \otimes 1)V^{\widehat{\mathbb{H}*}}, \quad \hat{f} \in L^1(\widehat{\mathbb{G}}), \quad T \in \mathcal{B}(L^2(\mathbb{H})),$$

where $V^{\widehat{\mathbb{H}}} \in L^\infty(\mathbb{H})' \overline{\otimes} L^\infty(\widehat{\mathbb{H}})$ is the right fundamental unitary of $\widehat{\mathbb{H}}$. Clearly, $L^\infty(\widehat{\mathbb{H}})$ is an $L^1(\widehat{\mathbb{G}})$ -submodule of $\mathcal{B}(L^2(\mathbb{H}))$ and $\gamma : L^\infty(\widehat{\mathbb{H}}) \rightarrow L^\infty(\widehat{\mathbb{G}})$ is a left $L^1(\widehat{\mathbb{G}})$ -module map. Thus, we may extend γ to a completely contractive left $L^1(\widehat{\mathbb{G}})$ -module map $\tilde{\gamma} : \mathcal{B}(L^2(\mathbb{H})) \rightarrow L^\infty(\widehat{\mathbb{G}})$. Then $\tilde{\gamma}$ is a unital complete contraction and therefore completely positive. Moreover, $L^\infty(\widehat{\mathbb{H}})$ is contained in the multiplicative domain of $\tilde{\gamma}$ (as it extends a $*$ -homomorphism), so the bimodule property of completely positive maps over their multiplicative domains (cf. [2]) ensures that

$$\tilde{\gamma}(\hat{x}T\hat{y}) = \gamma(\hat{x})\tilde{\gamma}(T)\gamma(\hat{y}), \quad \hat{x}, \hat{y} \in L^\infty(\widehat{\mathbb{H}}), \quad T \in \mathcal{B}(L^2(\mathbb{H})).$$

Thus, $\tilde{\gamma}$ is a completely bounded $L^\infty(\widehat{\mathbb{H}})$ - $\gamma(L^\infty(\widehat{\mathbb{H}}))$ bimodule map between $\mathcal{B}(L^2(\mathbb{H}))$ and $\mathcal{B}(L^2(\mathbb{G}))$. By [11, Theorem 2.5] we can approximate $\tilde{\gamma}$ in the point weak* topology by a net $\tilde{\gamma}_i$ of normal completely bounded $L^\infty(\widehat{\mathbb{H}})$ - $\gamma(L^\infty(\widehat{\mathbb{H}}))$ bimodule maps. Since $L^\infty(\mathbb{H})$ is standardly represented on $L^2(\mathbb{H})$, for each $g \in L^1(\mathbb{H})$ there exist $\xi, \eta \in L^2(\mathbb{H})$ such that $g = \omega_{\xi, \eta}|_{L^\infty(\mathbb{H})}$. Taking an orthonormal basis (e_j) of $L^2(\mathbb{H})$ it follows that

$$T \triangleleft g = (\omega_{\xi, \eta} \otimes \text{id})W^{\mathbb{H}*}(1 \otimes T)W^{\mathbb{H}} = \sum_j (\omega_{e_j, \eta} \otimes \text{id})(W^{\mathbb{H}*})T(\omega_{\xi, e_j} \otimes \text{id})(W^{\mathbb{H}}),$$

for $T \in \mathcal{B}(L^2(\mathbb{H}))$, where the series converges in the weak* topology. Thus,

$$\begin{aligned}
\tilde{\gamma}(T \triangleleft_{\mathbb{H}} g) &= \lim_i \tilde{\gamma}_i(T \triangleleft_{\mathbb{H}} g) \\
&= \lim_i \tilde{\gamma}_i \left(\sum_j (\omega_{e_j, \eta} \otimes \text{id})(W^{\mathbb{H}^*}) T (\omega_{\xi, e_j} \otimes \text{id})(W^{\mathbb{H}}) \right) \\
&= \lim_i \sum_j \tilde{\gamma}_i((\omega_{e_j, \eta} \otimes \text{id})(W^{\mathbb{H}^*}) T (\omega_{\xi, e_j} \otimes \text{id})(W^{\mathbb{H}})) \\
&= \lim_i \sum_j \gamma((\omega_{e_j, \eta} \otimes \text{id})(W^{\mathbb{H}^*})) \tilde{\gamma}_i(T) \gamma((\omega_{\xi, e_j} \otimes \text{id})(W^{\mathbb{H}})) \\
&= \lim_i \sum_j (\omega_{e_j, \eta} \otimes \text{id})((\text{id} \otimes \gamma)(W^{\mathbb{H}^*})) \tilde{\gamma}_i(T) (\omega_{\xi, e_j} \otimes \text{id})((\text{id} \otimes \gamma)(W^{\mathbb{H}})) \\
&= \lim_i (\omega_{\xi, \eta} \otimes \text{id})(\text{id} \otimes \gamma)(W^{\mathbb{H}})^* (1 \otimes \tilde{\gamma}_i(T)) (\text{id} \otimes \gamma)(W^{\mathbb{H}}) \\
&= \lim_i (g \otimes \text{id}) \alpha^l(\tilde{\gamma}_i(T)) \\
&= (g \otimes \text{id}) \alpha^l(\tilde{\gamma}(T)) \\
&= \tilde{\gamma}(T) \triangleleft_{\mathbb{H}} g.
\end{aligned}$$

Thus, $\tilde{\gamma}$ is a right $L^1(\mathbb{H})$ -module map.

Since $\tilde{\gamma}$ is also a left $L^1(\widehat{\mathbb{G}})$ -module map, for $x \in L^\infty(\mathbb{H})$ and $\hat{f}, \hat{g} \in L^1(\widehat{\mathbb{G}})$ we have

$$\begin{aligned}
\langle \Gamma_{\widehat{\mathbb{G}}}(\tilde{\gamma}(x)), \hat{f} \otimes \hat{g} \rangle &= \langle \tilde{\gamma}(x), \hat{f} \star_{\widehat{\mathbb{G}}} \hat{g} \rangle = \langle \hat{g} \triangleright_{\widehat{\mathbb{G}}} \tilde{\gamma}(x), \hat{f} \rangle \\
&= \langle \tilde{\gamma}(\hat{g} \triangleright_{\widehat{\mathbb{G}}} x), \hat{f} \rangle = \langle \hat{g}, 1 \rangle \langle \tilde{\gamma}(x), \hat{f} \rangle \\
&= \langle \tilde{\gamma}(x) \otimes 1, \hat{f} \otimes \hat{g} \rangle.
\end{aligned}$$

The standard argument then shows that $\tilde{\gamma}(x) \in L^\infty(\widehat{\mathbb{G}}) \cap L^\infty(\mathbb{G}) = \mathbb{C}1$. Hence, $\tilde{\gamma}|_{L^\infty(\mathbb{H})}$ defines a left invariant mean on $L^\infty(\mathbb{H})$, and \mathbb{H} is amenable. \square

Remark 3.3. Theorem 3.2 yields a new proof of the classical fact that amenability of a locally compact group passes to closed subgroups *without* the use of Bruhat functions or Leptin's theorem [18] on the equivalence of amenability of G and the existence of a bounded approximate identity in the Fourier algebra $A(G)$.

For a locally compact quantum group \mathbb{G} , the set of unitary elements $u \in L^\infty(\mathbb{G})$ satisfying $\Gamma(u) = u \otimes u$ forms a locally compact group under the relative weak* topology, called the *intrinsic group* of \mathbb{G} , and is denoted $\text{Gr}(\mathbb{G})$. The *character group* of \mathbb{G} is defined as $\widetilde{\mathbb{G}} := \text{Gr}(\widehat{\mathbb{G}})$. It follows that $\widetilde{\mathbb{G}}_a \cong G$, its underlying locally compact group, and $\widetilde{\mathbb{G}}_s \cong \widehat{G}$, the group of continuous characters on G . For more details on the character group and properties of the assignment $\mathbb{G} \mapsto \widetilde{\mathbb{G}}$ we refer the reader to [6, 14]. Since \mathbb{G} is always a closed quantum subgroup of \mathbb{G} [6, Theorem 5.5], we obtain the following generalization of [14, Theorem 5.14] beyond discrete quantum groups. We remark that the same conclusion was obtained in [6, Theorem 5.6] under the a priori stronger hypothesis that $\widehat{\mathbb{G}}$ is co-amenable.

Corollary 3.4. *Let \mathbb{G} be a locally compact quantum group. If \mathbb{G} is amenable then $\widetilde{\mathbb{G}}$ is amenable.*

We now study the existence of completely contractive $L^1(\widehat{\mathbb{G}})$ -module projections $P : L^\infty(\widehat{\mathbb{G}}) \rightarrow L^\infty(\widehat{\mathbb{H}})$. Our main tool is the following refinement of [3, Theorem 5.1].

Theorem 3.5. *Let \mathbb{G} and \mathbb{H} be locally compact quantum groups such that $\widehat{\mathbb{G}}$ is amenable and \mathbb{H} is a closed quantum subgroup of \mathbb{G} . Then \mathbb{H} is amenable if and only if $L^\infty(\widehat{\mathbb{H}})$ is 1-injective in $\mathbf{mod} - L^1(\widehat{\mathbb{G}})$.*

Proof. The canonical faithful right $L^1(\widehat{\mathbb{G}})$ -module action on $L^\infty(\widehat{\mathbb{H}})$ is given by

$$\hat{x} \star_{\widehat{\mathbb{G}}} \hat{f} = \hat{x} \star_{\widehat{\mathbb{H}}} \gamma_*(\hat{f}), \quad \hat{x} \in L^\infty(\widehat{\mathbb{H}}), \quad \hat{f} \in L^1(\widehat{\mathbb{G}}).$$

The embedding $\Delta : L^\infty(\widehat{\mathbb{H}}) \rightarrow \mathcal{CB}(L^1(\widehat{\mathbb{G}}), L^\infty(\widehat{\mathbb{H}}))$ is then

$$\Delta(\hat{x})(\hat{f}) = \hat{x} \star_{\widehat{\mathbb{G}}} \hat{f}, \quad \hat{x} \in L^\infty(\widehat{\mathbb{H}}), \quad \hat{f} \in L^1(\widehat{\mathbb{G}}).$$

Under the canonical identification $\mathcal{CB}(L^1(\widehat{\mathbb{G}}), L^\infty(\widehat{\mathbb{H}})) = L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}})$, one easily sees that $\Delta = (\gamma \otimes \text{id}) \circ \Gamma_{\widehat{\mathbb{G}}}$ and that the pertinent $L^1(\widehat{\mathbb{G}})$ -module structure on $L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}})$ is

$$X \star_{\widehat{\mathbb{G}}} \hat{f} = (\hat{f} \otimes \text{id} \otimes \text{id})(\Gamma_{\widehat{\mathbb{G}}} \otimes \text{id})(X), \quad X \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}}), \quad \hat{f} \in L^1(\widehat{\mathbb{G}}).$$

Let $U := (\gamma \otimes \text{id})(W^{\widehat{\mathbb{H}}}) \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}})$, where $W^{\widehat{\mathbb{H}}}$ is the left fundamental unitary of $\widehat{\mathbb{H}}$. The pentagonal relation (2.1) together with equation (3.1) imply

$$(3.3) \quad U_{23} W_{12}^{\widehat{\mathbb{G}}} = U_{13}^* W_{12}^{\widehat{\mathbb{G}}} U_{23},$$

where $W^{\widehat{\mathbb{G}}}$ is the left fundamental unitary of $\widehat{\mathbb{G}}$. By [3, Theorem 5.1], amenability of \mathbb{H} entails the existence of a completely contractive right $L^1(\widehat{\mathbb{H}})$ -module projection $E : \mathcal{B}(L^2(\mathbb{H})) \rightarrow L^\infty(\widehat{\mathbb{H}})$ with respect to the action

$$T \triangleleft_{\widehat{\mathbb{H}}} \hat{f} = (\hat{f} \otimes \text{id}) W^{\widehat{\mathbb{H}}} (1 \otimes T) W^{\widehat{\mathbb{H}}}, \quad \hat{f} \in L^1(\widehat{\mathbb{G}}), \quad T \in \mathcal{B}(L^2(\mathbb{H})).$$

Let $m \in L^\infty(\widehat{\mathbb{G}})^*$ be a left invariant mean, and define $\Phi : L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}}) \rightarrow L^\infty(\widehat{\mathbb{H}})$ by

$$\Phi(X) = E((m \otimes \text{id})(UXU^*)), \quad X \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}}).$$

Then for $\hat{x} \in L^\infty(\widehat{\mathbb{H}})$ we have

$$\Phi(\Delta(\hat{x})) = E((m \otimes \text{id})((\gamma \otimes \text{id})(W^{\widehat{\mathbb{H}}} \Gamma_{\widehat{\mathbb{H}}}(\hat{x}) W^{\widehat{\mathbb{H}}*}))) = E((m \otimes \text{id})(1 \otimes \hat{x})) = \hat{x},$$

so that Φ is a completely contractive left inverse to Δ . Now, fix $\hat{f} \in L^1(\widehat{\mathbb{G}})$ and $X \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{H}})$. Then

$$\begin{aligned} \Phi(X \star_{\widehat{\mathbb{G}}} \hat{f}) &= E((m \otimes \text{id})(U((\hat{f} \otimes \text{id} \otimes \text{id})(\Gamma_{\widehat{\mathbb{G}}} \otimes \text{id})(X))U^*)) \\ &= E((m \otimes \text{id})(U((\hat{f} \otimes \text{id} \otimes \text{id})(W_{12}^{\widehat{\mathbb{G}}} X_{23} W_{12}^{\widehat{\mathbb{G}}}))U^*)) \\ &= E((m \otimes \text{id})(\hat{f} \otimes \text{id} \otimes \text{id})(U_{23} W_{12}^{\widehat{\mathbb{G}}} X_{23} W_{12}^{\widehat{\mathbb{G}}} U_{23}^*)) \\ &= E((m \otimes \text{id})(\hat{f} \otimes \text{id} \otimes \text{id})(U_{13}^* W_{12}^{\widehat{\mathbb{G}}} U_{23} X_{23} U_{23}^* W_{12}^{\widehat{\mathbb{G}}} U_{13})) \quad (\text{by (3.3)}) \\ &= E((\hat{f} \otimes \text{id})(\text{id} \otimes m \otimes \text{id})(U_{13}^* W_{12}^{\widehat{\mathbb{G}}} U_{23} X_{23} U_{23}^* W_{12}^{\widehat{\mathbb{G}}} U_{13})) \\ &= E((\hat{f} \otimes \text{id})(U^*((\text{id} \otimes m \otimes \text{id})(W_{12}^{\widehat{\mathbb{G}}} (1 \otimes UXU^*) W_{12}^{\widehat{\mathbb{G}}}))U)). \end{aligned}$$

Using the fact that m is a left invariant mean on $L^\infty(\widehat{\mathbb{G}})$, it follows as in [4, Theorem 5.5] that

$$(\text{id} \otimes m \otimes \text{id})(W_{12}^{\widehat{\mathbb{G}}} (1 \otimes UXU^*) W_{12}^{\widehat{\mathbb{G}}}) = (\text{id} \otimes m \otimes \text{id})(1 \otimes UXU^*).$$

Thus,

$$\begin{aligned} \Phi(X \star_{\widehat{\mathbb{G}}} \hat{f}) &= E((\hat{f} \otimes \text{id})(U^*((\text{id} \otimes m \otimes \text{id})(1 \otimes UXU^*))U)) \\ &= E((\hat{f} \otimes \text{id})(\gamma \otimes \text{id})(W^{\widehat{\mathbb{H}}} (1 \otimes (m \otimes \text{id})(UXU^*))(\gamma \otimes \text{id})(W^{\widehat{\mathbb{H}}})) \\ &= E((\hat{f} \otimes \text{id})(\gamma \otimes \text{id})(W^{\widehat{\mathbb{H}}} (1 \otimes (m \otimes \text{id})(UXU^*))W^{\widehat{\mathbb{H}}})) \\ &= E((\gamma_*(\hat{f}) \otimes \text{id})(W^{\widehat{\mathbb{H}}} (1 \otimes (m \otimes \text{id})(UXU^*))W^{\widehat{\mathbb{H}}})) \\ &= E(((m \otimes \text{id})(UXU^*)) \triangleleft_{\widehat{\mathbb{H}}} \gamma_*(\hat{f})) \\ &= E((m \otimes \text{id})(UXU^*)) \star_{\widehat{\mathbb{H}}} \gamma_*(\hat{f}) \\ &= \Phi(X) \star_{\widehat{\mathbb{H}}} \gamma_*(\hat{f}) \\ &= \Phi(X) \star_{\widehat{\mathbb{G}}} \hat{f}. \end{aligned}$$

Hence, Φ is a right $L^1(\widehat{\mathbb{G}})$ -module map, and it follows that $L^\infty(\widehat{\mathbb{H}})$ is relatively 1-injective in $\mathbf{mod} - L^1(\widehat{\mathbb{G}})$. Since $L^\infty(\widehat{\mathbb{H}})$ is a 1-injective operator space, [3, Proposition 2.3] entails the 1-injectivity of $L^\infty(\widehat{\mathbb{H}})$ in $\mathbf{mod} - L^1(\widehat{\mathbb{G}})$.

Conversely, if $L^\infty(\widehat{\mathbb{H}})$ is 1-injective in $\mathbf{mod} - L^1(\widehat{\mathbb{G}})$ there exists a completely contractive right $L^1(\widehat{\mathbb{G}})$ -module projection $E : \mathcal{B}(L^2(\mathbb{H})) \rightarrow L^\infty(\widehat{\mathbb{H}})$. Then E is a right $L^1(\widehat{\mathbb{H}})$ -module projection, which by [3, Proposition 4.7] entails the amenability of \mathbb{H} . \square

Corollary 3.6. *Let \mathbb{G} and \mathbb{H} be locally compact quantum groups such that $\widehat{\mathbb{G}}$ is amenable and \mathbb{H} is a closed amenable quantum subgroup of \mathbb{G} in the sense of Vaes. Then there exists a completely contractive right $L^1(\widehat{\mathbb{G}})$ -module projection $P : L^\infty(\widehat{\mathbb{G}}) \rightarrow L^\infty(\widehat{\mathbb{H}})$.*

Proof. By Theorem 3.5, $L^\infty(\widehat{\mathbb{H}})$ is 1-injective as a right $L^1(\widehat{\mathbb{G}})$ -module. The identity $\text{id}_{L^\infty(\widehat{\mathbb{H}})}$ therefore extends to a completely contractive morphism $P : L^\infty(\widehat{\mathbb{G}}) \rightarrow L^\infty(\widehat{\mathbb{H}})$. \square

Since every co-commutative quantum group is amenable, the following corollary is immediate.

Corollary 3.7. *Let G be a locally compact group and let H be a closed amenable subgroup of G . Then there exists a completely contractive $A(G)$ -module projection $P : VN(G) \rightarrow VN(H)$.*

Remark 3.8. The existence of a completely bounded $A(G)$ -module projection $P : VN(G) \rightarrow VN(H)$ was shown in [12, Theorem 1.3] under the stronger assumption that G is amenable.

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